ON CERTAIN HOMOTOPY PROPERTIES OF SOME SPACES OF LINEAR AND PIECEWISE LINEAR HOMEOMORPHISMS. I ([1])

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ABSTRACT. Let K be a proper rectilinear triangulation of a 2-simplex S in the plane and $L(K)$ be the space of all homeomorphisms of S which are linear on each simplex of K and are fixed on Bd (S). The author shows in this paper that $L(K)$ with the compact open topology is simply-connected. This is a generalization of a result of S. S. Cairns in 1944 that the space $L(K)$ is pathwise connected. Both results will be used in Part II of this paper to show that $\pi_0(L_2) = \pi_1(L_2) = 0$ where L_n is a space of p.l. homeomorphisms of an n-simplex, a space introduced by R. Thom in his study of the smoothings of combinatorial manifolds.

In his study of the smoothings of Brouwer manifolds, S. S. Cairns considered a space of triangulations of a 2-simplex. He showed that two isomorphic (rectilinear) triangulations of a 2-simplex S, each having only three vertices on Bd (S), may be deformed continuously onto each other [1], [2]. To be more precise, we shall let S be a fixed 2-simplex in the Euclidean plane. For a simplicial subdivision K of S, let $L(K)$ be the space of all homeomorphisms from S onto S which are linear on each simplex of K and are pointwise fixed on Bd (S). We shall give $L(K)$ the compact open topology. Furthermore, a simplicial subdivision K of S is called a proper subdivision if K has only three vertices on Bd (S). Cairns' theorem may then be stated as follows: If K is a proper subdivision of S, the space $L(K)$ is path-connected. In this part of the paper, we shall prove the following.

Main theorem. *For any proper subdivision K of S, the space $L(K)$ is simply connected.*

Using this theorem, we shall show in Part II [5] that $\pi_0(L_2) = \pi_1(L_2) = 0$ where L_n is a space of p.l. homeomorphisms on an n-simplex, the space introduced by R. Thom [7] and studied by N. H. Kuiper [6] in connection with the

smoothing problems of combinatorial manifolds. The homotopy groups $\pi_k(L_n)$ are defined to be the inductive limit of $\pi_k(L(K))$'s with respect to the directed system of all subdivisions K of the n-simplex.

We shall prove our main theorem by inductive arguments on the number of vertices of K. In §1, we shall collect some of the necessary preliminaries. In §2, we shall describe a process, by which a space $L(K)$ may be related to a collection of spaces $\{L(K_i)\}$ where each K_i is a subdivision of S with a fewer number of vertices than K. This makes the induction possible. In §3, we shall show that by an inductive argument our main theorem may be reduced to a simpler statement—instead of deforming loops in $L(K)$, we need only establish the existence of certain paths in $L(K)$. The reduced statement will be proven in §4 by another induction.

1. **Preliminaries.** In the following, we shall let S be a fixed rectilinear 2-simplex in the Euclidean plane. All subdivisions of S are simplicial. If K is a subdivision of S and v a vertex of K, we shall let $\mathrm{St}(v, K)$ and $\mathrm{Lk}(v, K)$, or simply $\mathrm{St}(v)$ and $\mathrm{Lk}(v)$ denote the star and the link of v in K respectively. A subdivision K of S will be called *proper* if it has only three vertices on $\mathrm{Bd}(S)$. If K is a subdivision of S, a *linear homeomorphism* from K to S is a homeomorphism from the space S into itself whose restriction to each simplex of K is affinely linear. Clearly each linear homeomorphism from K to S is completely determined by its image of the vertices of K. We shall use $L(K)$ to denote the space of all linear homeomorphisms from K to S which are pointwise fixed on $\mathrm{Bd}(S)$ together with the compact open topology. Note that the space $L(K)$ is metrizable, say with a metric $\gamma(f, g) = \max\{d(f(v), g(v)) \mid v$ an inner vertex of $K\}$ where an *inner* vertex of K is a vertex of K which is not on $\mathrm{Bd}(S)$ and the metric d is the sup-metric of the Euclidean plane, i.e., $d((x_1, x_2), (y_1, y_2)) = \max_{i=1,2} |x_i - y_i|$.

We also note that for each subdivision K of S, the space $L(K)$ may be identified with a subset of R^n for some n. This can be seen as follows, since each element of $L(K)$ is completely determined by its image of the inner vertices of K. If an ordering is assigned to the inner vertices, say $v_1, v_2, \cdots, v_k$ of K, each $f \in L(K)$ may then be identified by the point $(f(v_1), f(v_2), \cdots, f(v_k)) \in R^2 \times R^2 \times \cdots \times R^2$ (k copies). Since K has only finitely many inner vertices, it can be shown that if the position of $f(v_i)$ for each inner vertex v_i of K is given a sufficiently small perturbation, the resulting linear map will still be injective on each star of the vertices of K, hence, will be an element of $L(K)$ (cf. Lemma 2.1 of [5]). This shows that the space $L(K)$ may in fact be identified with an open subset of R^{2k}. We may therefore consider $L(K)$ as an open submanifold of some Euclidean space.

We now list some of the basic properties of the spaces $L(K)$ which will be used later.

Remark 1.1. Observe that if K' is a subdivision of K, then $L(K) \subset L(K')$. The inclusion map is a homeomorphism of $L(K)$ into $L(K')$.

Remark 1.2. For each $f \in L(K)$, the set $\{f(\sigma) |\ \sigma \in K\}$ also forms a subdivision of S, which will be denoted by $f(K)$. It is easy to see that for each $f \in L(K)$, the spaces $L(K)$ and $L(f(K))$ are homeomorphic (the map: $L(f(K)) \rightarrow L(K)$ carrying each g of $L(f(K))$ onto gf in $L(K)$ is clearly a homeomorphism).

Definition 1.3. A subdivision K of S is said to be *decomposable* if there are vertices v_a, v_b, v_c of K such that

1. At least one of these three vertices is an inner vertex of K.

2. The 1-simplices $\langle v_a, v_b \rangle, \langle v_b, v_c \rangle$ and $\langle v_c, v_a \rangle$ all belong to K but the 2-simplex $\langle v_a, v_b, v_c \rangle$ does not belong to K.

In this case, the triangle $\langle v_a, v_b, v_c \rangle$ is called a *pivot triangle* of K.

Proposition 1.4. *If K is a decomposable subdivision of S, $L(K)$ is homeomorphic to a Cartesian product $L(K_1) \times L(K_2)$ where K_1 and K_2 are both subdivisions of S with a fewer number of vertices than K. In fact, the number of vertices of K equals the sum of those of K_1 and K_2.*

Proof. If K is decomposable with a pivot triangle $\langle v_a, v_b, v_c \rangle$, we shall let K_1 be the subdivision of S which agrees with K outside the region $\langle v_a, v_b, v_c \rangle$ and on that region, we require $\langle v_a, v_b, v_c \rangle$ to be a 2-simplex of K_1. Let K_2 be a subdivision of S which is isomorphic to the subdivision given by K on the triangle $\langle v_a, v_b, v_c \rangle$. Clearly, $L(K)$ is homeomorphic to $L(K_1) \times L(K_2)$ and if k, k_1 and k_2 represent the number of inner vertices of K, K_1 and K_2 respectively, then k, k_1, $k_2 > 0$ and $k = k_1 + k_2$.

On the other hand, if the subdivision K is indecomposable, we shall show in the remaining part of this section that the space $L(K)$ must contain certain "nice" elements. We shall call a polyhedral disk *strictly convex* if it is convex and no three of the vertices on its boundary lie on a straight line.

Proposition 1.5. *Let K be a proper indecomposable subdivision of S, for each inner vertex v_0 of K, there is an $f \in L(K)$ such that $\mathrm{St}(f(v_0), f(K))$ is strictly convex.*

Proof. Let $H = K - \{\sigma \in K |\ v_0$ is a vertex of $\sigma\}$. Note that $S - |H| = \mathrm{St}(v_0, K)$. We want to find a linear homeomorphism f from H into S such that $f | \mathrm{Bd}(S) = $ identity and $S - f(|H|)$ is strictly convex. Such an f will be defined inductively. First, let $H_0 = \{\sigma \in K |\ \sigma$ lies on $\mathrm{Bd}(S)\}$ and $f_0: H_0 \rightarrow S$ be the identity map on H_0. Suppose we have already defined a subcomplex H_n of H

containing H_0 and a linear homeomorphism f_n from H_n into S such that

1. For each simplex σ of K, if all the vertices of σ belong to H_n, then σ must itself be a simplex of H_n.

2. $f_n \mid H_0$ = identity map.

3. Each component of $S - f_n(\mid H_n\mid)$ is strictly convex.

If there is a vertex v of H_n which lies on $\text{Bd}(S - \mid H_n\mid)$ but not on $\text{Bd}(S - \mid H\mid) = \mid \text{Lk}(v_0, K)\mid$, we may always construct a subcomplex H_{n+1} of H strictly containing H_n and a linear homeomorphism f_{n+1} from H_{n+1} into S such that the conditions listed above are all satisfied if n is replaced by $n+1$. We first define a complex $H_n' = H_n \cup \{\sigma \in K \mid \sigma$ lies on $\overline{\text{St}}(v, K)\}$, then let $H_{n+1} = \{\sigma \in K\mid$ all vertices of σ are in $H_n'\}$. Note that H_{n+1} is a subcomplex of H strictly containing H_n.

We now define f_{n+1} by first defining a linear map $f_n': H_n' \to S$ as follows: let $f_n' \mid H_n = f_n$. As for $H_n' - H_n$, note that the component of $S - f_n(\mid H_n\mid)$ which has $f_n(v)$ as a vertex on its boundary is a strictly convex polyhedral disk. Depending on the situation, we shall define f_n' in one of the following manners: adding the remaining part of $\overline{\text{St}}(v, K)$ inside and along the edges of that convex disk, or connecting $\overline{\text{St}}(v, K)$ to some other vertices on the boundary of that disk and thus cutting the disk into two or more convex regions, or filling up the whole disk. In each case, we may define an injective linear map f_n' from H_n' into S in such a way that each component of $S - f_n'(\mid H_n\mid)$ is a strictly convex disk. Since each simplex σ of H_{n+1} has all its vertices in H_n', f_n' extends uniquely to a linear map $f_{n+1}: H_{n+1} \to S$. Note that the map f_{n+1} is still injective. Each component of $S - f_{n+1}(\mid H_{n+1}\mid)$ is still strictly convex, for the extension only fills up certain components of $S - f_n'(\mid H_n'\mid)$, or cuts some component into two or more convex regions.

Continuing this process, we finally obtain a complex $H_m \subset H$ and a linear homeomorphism $f_m: H_m \to S$ satisfying the three conditions listed above together with the condition:

4. Each vertex of H_m which lies on $\text{Bd}(S - \mid H_m\mid)$ also lies on $\text{Bd}(S - \mid H\mid)$ $= \mid\text{Lk}(v_0, K)\mid$.

We now show that under these conditions $H_m = H$. First observe that condition 4 implies that $S - \mid H_m\mid$ has only one component. Then, note that each vertex of $\text{Lk}(v_0, K)$ must also lie on $\text{Bd}(S - \mid H_m\mid)$, for otherwise, there must be two vertices $v_a, v_b \in H_m$ which are consecutive on $\text{Bd}(S - \mid H_m\mid)$ but not consecutive on $\text{Lk}(v_0, K)$. Therefore, $\langle v_0, v_a, v_b \rangle$ must be a pivot triangle. This contradicts the fact that K is indecomposable. Hence, $\text{Bd}(S - \mid H_m\mid) = \mid\text{Lk}(v_0, K)\mid$ and H_m must contain all the simplices $\sigma \in K$ which does not have v_0 as a vertex.

Hence, $H_m = H$ and $f_m: H \to S$ is clearly a desired map.

Using this method, we may in fact prove a stronger statement. Roughly speaking, for two suitably chosen inner vertices v_0 and v_1 of an indecomposable subdivision K, we may find a map $f \in L(K)$ such that both $\mathrm{St}(f(v_0), f(K))$ and $\mathrm{St}(f(v_1), f(K))$ are strictly convex. There are certain restrictions on the vertices v_0 and v_1 for the existence of such a map $f \in L(K)$. We need the following concept to formulate these restrictions.

Definition 1.6. Let K be a subdivision of S. Two vertices v_a, v_b of K such that $v_a \notin \overline{\mathrm{St}}(v_b, K)$ are called a *binding pair* if one of the following holds.

1. $\mathrm{Lk}(v_a) \cap \mathrm{Lk}(v_b)$ is disconnected.
2. $\mathrm{Lk}(v_a) \cap \mathrm{Lk}(v_b)$ is connected but contains three or more vertices of K.

Otherwise (i.e., when both 1 and 2 are false), these two vertices are said to be a *nonbinding pair*.

We note that if v_a, v_b are a binding pair of the subdivision K, it is impossible to find a map f in $L(K)$ such that both $\mathrm{St}(f(v_a), f(K))$ and $\mathrm{St}(f(v_b), f(K))$ are strictly convex, for the strict convexity of one of these regions clearly forces the other to "cave in." It turns out that this is the only restriction for the existence of such a map. In fact, we have the following

Proposition 1.7. *Let K be a proper indecomposable subdivision of S, v_0, v_1 be inner vertices of K such that $v_1 \notin \overline{\mathrm{St}}(v_0)$ and v_0, v_1 are a nonbinding pair. One may always find a map $f \in L(K)$ such that both $\mathrm{St}(f(v_0), f(K))$ and $\mathrm{St}(f(v_1), f(K))$ are strictly convex.*

We shall outline a proof of this proposition. Suppose K, v_0, v_1 satisfy the hypothesis of the proposition. Let $H = K - \{\sigma \in K | \ \sigma$ has either v_0 or v_1 as a vertex$\}$. We therefore want to find a linear homeomorphism f from H into S such that $f | \mathrm{Bd}(S) =$ identity and $S - f(|H|)$ consists of two strictly convex regions.

By the same argument as in the proof of 1.5, we may construct inductively a subcomplex H_m of H and a linear homeomorphism $f_m : H_m \to S$ such that

1. For each simplex σ of K, if all the vertices of σ belong to H_m, then σ must itself be a simplex of H_m.
2. $\mathrm{Bd}(S) \subset |H_m|$ and $f_m | \mathrm{Bd}(S) =$ identity.
3. Each component of $S - f_m(|H_m|)$ is strictly convex.
4. Each vertex of H_m which lies on $\mathrm{Bd}(S - |H_m|)$ also lies on $\mathrm{Bd}(S - |H|)$ $= |\mathrm{Lk}(v_0, K)| \cup |\mathrm{Lk}(v_1, K)|$.

By condition 4, one sees that $S - |H_m|$ can have only one or two components. Suppose there are two components. Since $(S - |H|) \subset (S - |H_m|)$, these two components must contain $\mathrm{St}(v_0, K)$ and $\mathrm{St}(v_1, K)$ respectively. Then by condition 4 again, we conclude that $H_m = H$, for otherwise, we can always construct a pivot triangle and get a contradiction (cf. the proof of 1.5). Hence, f_m gives rise to a desired map.

Suppose $S - |H_m|$ has exactly one component. Then $S - |H_m|$ is a polyhedral disk containing both $\mathrm{St}(v_0, K)$ and $\mathrm{St}(v_1, K)$. By condition 4 and the assumption that K is indecomposable, we see that $\mathrm{Bd}(S - |H_m|)$ intersects each of $\mathrm{Lk}(v_0, K)$ and $\mathrm{Lk}(v_1, K)$ in an arc. Using the fact that v_0, v_1 are a nonbinding pair and condition 4, we may define a map $f: H \to S$ from f_m by adding the image of those simplices in $H - H_m$ inside the convex disk $S - f_m(|H_m|)$ in such a way that the disk is cut into two convex regions. This then gives rise to a desired map. A detailed proof of this proposition (by a different method) can be found in [3]. It may also be proven in a more general setting [4].

2. **The reducibility of the space $L(K)$.** Let K be a subdivision of S. In this section, we shall first show that for each inner vertex v_i of K, a subspace $L_{v_i}(K)$ of $L(K)$ may be defined such that $L_{v_i}(K)$ and $L(K)$ have the same homotopy type. We shall then show that for a suitably chosen vertex v_i, the space $L_{v_i}(K)$ may be considered as the union of a finite collection of spaces $\{L(K_j)\}$ where each K_j is a subdivision with a fewer number of vertices than K. This makes it possible to adopt the induction argument in the proof of our main theorem.

Notation 2.1. Let P be a polygonal circle in R^2 (i.e., P is a simplicial complex in R^2 and $|P|$ is homeomorphic to the 1-sphere S^1). Let $[P]$ be the union of $|P|$ and the bounded component of $R^2 - |P|$. For each 1-simplex $\langle v_1, v_2 \rangle$ of P, we shall use $H_P\langle v_1, v_2 \rangle$ to denote the open half-plane of R^2 such that

   1. $\langle v_1, v_2 \rangle$ lies on $\mathrm{Bd}(H_P\langle v_1, v_2 \rangle)$.
   2. $\bar{H}_P\langle v_1, v_2 \rangle$ (the closed half-plane) contains a neighbourhood of $\langle v_1, v_2 \rangle$ in $[P]$.

Definition 2.2. Let P be a polygonal circle in R^2. We define the *core* of P, $\mathrm{cor}(P)$, to be the set $\mathrm{cor}(P) = \bigcap \{H_P\langle v_i, v_j \rangle | \langle v_i, v_j \rangle$ is a 1-simplex of $P\}$.

Remark 2.3. Observe that for a polygonal circle P in R^2, $\mathrm{cor}(P)$, being the intersection of a collection of convex sets, is convex. Furthermore, if $\mathrm{cor}(P) \neq \varnothing$, then for any point $y \in \mathrm{cor}(P)$, $y * P$, the join of y and P, forms a triangulation of the region $[P]$.

Remark 2.4. Suppose that v_0 is an inner vertex of a subdivision K and $f \in L(K)$. We note that $\mathrm{cor}(\mathrm{Lk}(f(v_0), f(K))) \neq \varnothing$, for $f(v_0)$ is itself a point of that set. We further note that if $g: K \to S$ is a linear map with $g(v) = f(v)$ for all vertices $v \neq v_0$ of K, then the necessary and sufficient condition for g to be an element of $L(K)$ is that $g(v_0) \in \mathrm{cor}(\mathrm{Lk}(f(v_0), f(K)))$.

Definition 2.5. Let K be a subdivision of S. For each inner vertex v_i of K, we shall let $L_{v_i}(K)$ or simply $L_i(K)$ be the subspace of $L(K)$ consisting of all elements $f \in L(K)$ such that $f(v_i)$ is located at the centroid of the set $\mathrm{cor}(\mathrm{Lk}(f(v_i), f(K)))$.

Proposition 2.6. *For each inner vertex v_0 of K, $L_0(K)$ is a deformation retract of $L(K)$; hence, $L_0(K)$ and $L(K)$ are of the same homotopy type.*

Proof. We shall define a deformation D of $L(K)$ onto $L_0(K)$ as follows. For each $f \in L(K)$, let w_f be the centroid of the bounded convex set $\mathrm{cor}\,(\mathrm{Lk}\,(f(v_0),\, f(K)))$. Now for each t in the unit interval and each $f \in L(K)$, we let $D(t, f)$ be the element of $L(K)$ such that, for each inner vertex v of K, $D(t, f)(v) = f(v)$ if $v \neq v_0$ and $D(t, f)(v_0) = tw_f + (1 - t)f(v_0)$. Since each element of $L(K)$ is completely determined by its image of the inner vertices of K and since, for each $f \in L(K)$, $\mathrm{cor}\,(\mathrm{Lk}\,(f(v),\, f(K)))$ is convex, $D(t, f)$ is indeed a well-defined element in $L(K)$ by 2.4. The continuity of D may be proved easily by using the metric on $L(K)$ given in §1.

We shall now show that for a sufficiently nice inner vertex v_0 of K, the space $L_0(K)$ may be split into a union of spaces $\{L(K_i)\}$ where each K_i is a subdivision with a fewer number of inner vertices than K.

Definition 2.7. Let P be a polygonal circle and v be a vertex of P. Using the notations of 2.1, we define the *wedge* at v with respect to P, $W_v(P)$, to be the set $W_v(P) = H_P\langle v, v_1\rangle \cup H_P\langle v, v_2\rangle$ where v_1 and v_2 are the two vertices which are next to v on P.

We also define the *complementary wedge* at v with respect to P, $CW_v(P)$, to be the set $CW_v(P) = \bigcap \{H_P\langle v_i, v_j\rangle \mid \langle v_i, v_j\rangle \in P$ but $v \notin \langle v_i, v_j\rangle\}$.

Remark 2.8. (a) Observe that for any polygonal circle P and any vertex v of P, $\mathrm{cor}\,(P) = W_v(P) \cap CW_v(P)$.

(b) If v is a vertex of P such that $v \in CW_v(P)$, then $P \cup \{v * \delta \mid \delta \in P$ but v is not a vertex of $\delta\}$ also forms a triangulation of the region $[P]$. This triangulation is said to be obtained by *starring v with respect to P.*

Definition 2.9. Let P be a polygonal circle. A vertex v of P is said to be *accessible* on P if $v \in CW_v(P)$.

Let K be a subdivision of S. Observe that for each inner vertex v_0 of K, $\mathrm{Lk}\,(v_0)$ is a polygonal circle. We may therefore speak of the accessibility on $\mathrm{Lk}\,(v_0)$ for vertices of $\mathrm{Lk}\,(v_0)$.

Remark 2.10. Let K be a subdivision of S and v be an inner vertex of K. Suppose that v_i is a vertex of $\mathrm{Lk}\,(v)$ accessible on $\mathrm{Lk}\,(v)$. Then by 2.9 and 2.8(b), we may obtain another subdivision K' of S from K as follows. Let K' agree with K outside the region $\mathrm{St}\,(v, K)$ and on that region, let K' be the triangulation obtained by starring v_i with respect to $\mathrm{Lk}\,(v, K)$.

K' is called a subdivision obtained from K by collapsing v to v_i. K' has one fewer vertex than K since the vertex v is no longer a vertex of K'. Also note that if K is a proper subdivision, so is K'.

Proposition 2.11. *If K' is the subdivision obtained from K by collapsing some vertex v_0 to an accessible vertex $v_i \in \mathrm{Lk}(v_0, K)$, the space $L(K')$ may be imbedded as an open subset of the space $L_0(K)$ such that, for an element $f \in L_0(K)$, f belongs to the image of the imbedding if and only if $f(v_i)$ is accessible on $\mathrm{Lk}(f(v_0), f(K))$.*

Proof. Let $P = \mathrm{Lk}(v_0, K)$. Note that P is also a subcomplex of K'. We claim that, for each $f' \in L(K')$, $\mathrm{cor}(f'(P)) \neq \emptyset$. To see this, we first note that for each 1-simplex $\langle v_a, v_b \rangle$ of P such that $v_i \neq v_a$ or v_b, $\langle v_i, v_a, v_b \rangle$ is a 2-simplex of K'. Hence, $f'(v_i) \in H_{f'(P)}\langle f'(v_a), f'(v_b) \rangle$. Since this is true for each such 1-simplex, $f'(v_i) \in CW_{f'(v_i)}(f'(P))$. Also note that $CW_{f'(v_i)}(f'(P))$ is an open set in the plane and $W_{f'(v_i)}(f'(P))$ contains points arbitrarily close to the point $f'(v_i)$, hence, $\mathrm{cor}(f'(P)) = W_{f'(v_i)}(f'(P)) \cap CW_{f'(v_i)}(f'(P)) \neq \emptyset$.

We can now define an imbedding of $L(K')$ into $L(K)$ by sending each $f' \in L(K')$ into the map $f \in L(K)$ where $f(w) = f'(w)$ for each vertex w of K such that $w \neq v_0$ and $f(v_0)$ lies at the centroid of $\mathrm{cor}(f'(P))$. The position of $f(v_0)$ ensures us that $f \in L_0(K)$.

This is indeed a well-defined map from $L(K')$ into $L_0(K)$ and it is injective, for the image of the vertices of a map determines the map completely. Using the metrics on $L(K)$ and $L(K')$ given in §1, one sees immediately that this map is in fact a homeomorphism into.

To see the image of this imbedding, we note that for any $f' \in L(K')$, $f'(v_i) \in CW_{f'(v_i)}(f'(P))$ since for each 1-simplex $\langle v_a, v_b \rangle$ of P such that $v_i \neq v_a$ or v_b, $\langle f'(v_i), f'(v_a), f'(v_c) \rangle$ is a 2-simplex of $f(K')$. Hence, $f'(v_i)$ is accessible on $f'(P)$. But the corresponding f under the imbedding agrees with f' entirely on P. Therefore, $f(v_i)$ is also accessible on $f(P) = \mathrm{Lk}(f(v_0), f(K))$. Conversely, if $f(v_i)$ is accessible on $\mathrm{Lk}(f(v_0), f(K))$, the region $\mathrm{St}(f(v_0), f(K))$ may be retriangulated by collapsing $f(v_0)$ to $f(v_i)$ so that it becomes the image of an element $f' \in L(K')$. Clearly, the imbedding carries f' to f.

To show that the image of $L(K')$ under the imbedding is open in $L_0(K)$, we let $f \in L_0(K)$ be a map such that $f(v_i)$ is accessible on $f(P)$. It is clear that for a map $g \in L_0(K)$ such that $g(v)$ is close enough to $f(v)$ for all vertices $v \in K$, $g(v_i)$ is also accessible on $g(P)$. This shows that any map g close enough to f in the metric space $L_0(K)$ is also in the image of $L(K')$. Hence, the image is open.

Remark 2.12. The imbedding of $L(K')$ into $L_0(K)$ described in the preceding proof will be called the *canonical imbedding*. In the following, if K' is a subdivision obtained from a subdivision K by collapsing some inner vertex v_0 of K, we shall consider $L(K')$ as a subset of $L_0(K)$ via the canonical imbedding of $L(K')$ into $L_0(K)$.

Before we may establish our condition on the reducibility of the space $L_0(K)$, we first need a lemma.

Lemma 2.13. *If P is a polygonal circle with 5 or fewer vertices, at least one of the vertices is accessible on P.*

Proof. The lemma is easily seen to be true if P has only 3 or 4 vertices, for P is then a nondegenerate triangle or a quadrangle (not necessarily convex). In either case P has at least one accessible vertex.

When P has 5 vertices, P is a pentagon. Since the sum of all the interior angles of a pentagon is 3π, the possible shapes of P may be divided into the following four cases: (1) All the interior angles of P are less than π. (2) Exactly one of the interior angles, say the angle at the vertex v_1, is greater than or equal to π. (3) Two nonconsecutive interior angles, say those at v_1 and v_3, are greater than or equal to π. (4) Two consecutive interior angles are greater than or equal to π. A picture in each case shows immediately that in case (1) all the vertices of P are accessible; in case (2), the vertex v_1 is always accessible; in case (3), one of the vertices v_1 and v_3 is accessible and finally in case (4), the (unique) vertex of P which is not next to any of the two consecutive vertices is always accessible.

Corollary 2.14. *For any polygonal circle P in R^2 with five or fewer vertices, $\text{cor}(P) \neq \varnothing$.*

Proof. We have just shown that for such a P, there is always a vertex v of P such that $v \in CW_v(P)$. But $CW_v(P)$ is an open set in R^2 and the set $W_v(P)$ contains points arbitrarily close to v, hence $\text{cor}(P) = W_v(P) \cap CW_v(P) \neq \varnothing$.

We may now establish our condition on the vertex v_0 so that the space $L_0(K)$ is reducible (see Proposition 2.16).

Definition 2.15. Let K be a subdivision of S. For each vertex v_i of K, we define the *incidence number* of v_i in K, denoted by $m(v_i)$ or simply m_i, to be the number of vertices in $\text{Lk}(v_i, K)$ (hence, also the number of 1-simplices of K incident on v_i).

Proposition 2.16. *Let K be a subdivision of S with n inner vertices. Let v_0 be an inner vertex of K such that*
    1. *the incidence number m_0 of $v_0 \leq 5$;*
    2. *$\text{St}(v_0, K)$ is strictly convex.*
Then $L_0(K)$ may be written as the union of a collection of open subspaces $\{L(K_i)\}$ where each K_i is a subdivision obtained from K by collapsing the vertex v_0 to a vertex on $\text{Lk}(v_0, K)$. (Hence, the collection $\{L(K_i)\}$ consists of at most five elements and each subdivision K_i has exactly $n-1$ vertices.)

Proof. Since $\mathrm{St}(v_0, K)$ is strictly convex, each vertex v_i on $\mathrm{Lk}(v_0)$ is accessible on $\mathrm{Lk}(v_0)$. By 2.10, a subdivision K_i may be formed for each $v_i \in \mathrm{Lk}(v_0)$ by collapsing v_0 to v_i. Proposition 2.11 shows that each space $L(K_i)$ may be considered as an open subset of $L_0(K)$ and that an element $f \in L_0(K)$ belongs to $L(K_i)$ if and only if $f(v_i)$ is also accessible on $\mathrm{Lk}(f(v_0), f(K))$. But 2.13 ensures us that for each map $f \in L_0(K)$, at least one of the vertices on $\mathrm{Lk}(f(v_0), f(K))$ is accessible. Hence the space $L_0(K)$ is completely covered by the collection $\{L(K_i)\}$.

We shall now show that for a proper subdivision K of S, there is at least one inner vertex with the incidence number ≤ 5. The proof depends on the following

Cominatorial Lemma 2.17. *If D is a polyhedral disk in R^2 (i.e., D is a simplicial complex imbedded in R^2 such that the underlying space $|D|$ is a topological disk), then*

$$\sum_{v_i = \text{inner vertex of } D} (6 - m_i) = 6 + \sum_{v_i \in \mathrm{Bd}(D)} (m_i - 4)$$

where for each vertex v_i, $m_i = $ number of 1-simplices of D incident on v_i.

Proof. Let s, t, n and k be the number of the 2-simplices, 1-simplices, inner vertices of D and vertices of D lying on $\mathrm{Bd}(D)$ respectively. We shall now establish the following relations:

1. $(n + k) - t + s = 1$.
2. $3s = 2t - k$.
3. $2t = \Sigma_{v_i \in D} \, m_i$.

The first equation is clear since the Euler characteristic of $|D|$ is 1. For the second equation, if we multiply the number of 2-simplices of D by 3, we are counting each of those 1-simplices of D lying on $\mathrm{Int}\,|D|$ twice and each of those lying on $\mathrm{Bd}(|D|)$ only once. But the number of 1-simplices of D lying on $\mathrm{Bd}(|D|)$ is also k. Hence, $3s = 2(t - k) + k = 2t - k$. The third equation is clear, for if we sum up the number of 1-simplices incident on v_i for each v_i, we are counting each 1-simplex of D exactly twice.

Now, from equations 1 and 2, we have

$$3(n + k) - 3t + (2t - k) = 3 \quad \text{or} \quad 3n + 2k - t = 3.$$

Multiplying it through by 2 and using equation 3, we have

$$6n + 4k - \sum_{v_i \in D} m_i = 6$$

or

$$\left(\sum_{v_i \in \text{Int} |D|} 6 \right) + \left(\sum_{v_i \in \text{Bd}(|D|)} 4 \right) - \left(\sum_{v_i \in D} m_i \right) = 6.$$

Hence,

$$\sum_{v_i \in \text{Int} |D|} (6 - m_i) = 6 + \sum_{v_i \in \text{Bd}(|D|)} (m_i - 4).$$

Corollary 2.18. *For each proper subdivision K of S with at least one inner vertex, there always exists an inner vertex v_0 of K with $m_0 \leq 5$.*

Proof. A proper subdivision K of S has only 3 vertices on $\text{Bd}(S)$. For each of these vertices $v_i \in \text{Bd}(S)$, $m_i \geq 3$ since K has at least one inner vertex. Hence, the combinatorial lemma above reduces to $\Sigma_{v_i \text{ inner}} (6 - m_i) \geq 3$. From which the corollary follows.

As an application of the machinery developed so far, we shall give another proof of the Cairns' result on the connectivity of the space $L(K)$ for a proper subdivision K of S [1], [2].

Proposition 2.19. *For each proper subdivision K of S, the space $L(K)$ is path-connected.*

Proof. We shall prove this by induction on the number of inner vertices of K. The proposition is trivially true if the number is one, for in that case $L(K)$ is homeomorphic to $\text{Int}(S)$. Now consider a proper subdivision K with n inner vertices. We may assume that the subdivision K is indecomposable, for otherwise by 1.4, the induction may be completed immediately. By 2.18, we may choose an inner vertex v_0 of K such that $m_0 \leq 5$. Using 1.5 and then 1.2 if necessary, we may assume that $\text{St}(v_0, K)$ is strictly convex. Now we may use 2.16 to write the corresponding space $L_0(K)$ into a union of at most five spaces $L(K_i)$ where each K_i is a proper subdivision of S with only $n - 1$ vertices.

We now claim that the intersection of these spaces is nonempty. Let e be the element of $L_0(K)$ such that $e(v) = v$ for each vertex $v \neq v_0$ of K and $e(v_0) = $ centroid of $\text{cor}(\text{Lk}(v_0))$, then every vertex on $\text{Lk}(e(v_0), e(K))$ is accessible since the set $\text{St}(e(v_0), e(K))$ is strictly convex. Therefore, e belongs to $L(K_i)$ for each K_i and hence $\bigcap_i L(K_i) \neq \emptyset$.

Applying the induction hypothesis on the spaces $L(K_i)$, we immediately get the path-connectedness of the space $L_0(K)$. But $L_0(K)$ and $L(K)$ are of the same homotopy type, hence, $L(K)$ is also path-connected.

3. A reduction of the main theorem. In this section we shall show that our main theorem may be reduced to a statement which asserts only the existence of certain paths in the space $L(K)$ for some special subdivisions K of S. The reduced statement may be stated in the following form.

Theorem A. *Let K be a proper subdivision of S, v_0 an inner vertex of K such that $m_0 = 4$ and $\mathrm{St}(v_0, K)$ is strictly convex. For each $f \in L(K)$ with $\mathrm{St}(f(v_0), f(K))$ strictly convex, there exists a path γ in $L(K)$ from the identity element $e \in L(K)$ to f such that, for each $t \in I$, $\mathrm{St}(\gamma_t(v_0), \gamma_t(K))$ is strictly convex.*

We now show that our main theorem may be proved by assuming the validity of Theorem A. Theorem A will then be proved in the next section.

We shall use induction on the number of inner vertices of K. If K has one inner vertex, $L(K)$ is clearly simply connected, for the space $L(K)$ is contractible. We now consider a proper subdivision K with n inner vertices. By 2.18, we may choose an inner vertex v_0 of K such that the incidence number $m_0 \leq 5$. Using 1.4, we may clearly assume K to be indecomposable. In particular, this shows $m_0 = 4$ or 5 since the incidence number of any inner vertex is clearly at least 3 and, if $m_0 = 3$, $\mathrm{Lk}(v_0, K)$ would have been a pivot triangle. Now applying 1.5 and 1.2 if necessary, we may assume that $\mathrm{St}(v_0, K)$ is strictly convex.

Since the space $L(K)$ and $L_0(K)$ are of the some homotopy type, we need only show the simple connectedness of $L_0(K)$. We shall take as the base point of $L_0(K)$ the element $e_0 \in L_0(K)$ where $e_0(v) = v$ for all inner vertices $v \neq v_0$ and $e_0(v_0) = $ the centroid of $\mathrm{car}(\mathrm{Lk}(v_0, K))$. By 2.16, $L_0(K)$ may be written as the union of a collection of open subsets $\{L(K_i)\}$ where each K_i is a proper subdivision obtained from K by collapsing the vertex v_0 to a vertex v_i in $\mathrm{Lk}(v_0, K)$. Note that $e_0 \in L(K_i)$ for each i.

We shall first consider the case when $m_0 = 4$. In this case the space $L_0(K)$ is covered by four path-connected (2.19) open subsets $\{L(K_i)\}_{i=1}^4$ with $e_0 \in \bigcap_{i=1}^4 L(K_i)$. Since each K_i is a proper subdivision with $n - 1$ vertices, each $L(K_i)$ is simply connected by the induction hypothesis. Hence, if we can further prove that the intersection of each pair $L(K_i) \cap L(K_j)$ is also path connected, applying the usual Seifert and Van Kampen theorem, we may immediately conclude the simple-connectedness of $L_0(K)$ and thus settle the case for $m_0 = 4$. The path-connectedness of each intersection is proven in the following lemma.

Lemma 3.1. *For each pair of distinct vertices v_i, v_j on $\mathrm{Lk}(v_0, K)$, $L(K_i) \cap L(K_j)$ is path-connected.*

Proof. Since $\mathrm{St}(v_0, K)$ is assumed to be strictly convex, by 2.11, an element $f \in L_0(K)$ belongs to a subspace $L(K_i)$ if and only if the vertex $f(v_i)$ is accessible on $\mathrm{Lk}(f(v_0), f(K))$. Let v_i, v_j be two distinct vertices on $\mathrm{Lk}(v_0, K)$. We now consider the intersection $L(K_i) \cap L(K_j)$.

First consider the case when v_i, v_j are two nonconsecutive vertices on

$Lk(v_0, K)$. Since $m_0 = 4$, $Lk(v_0, K)$ is a quadrilateral. Observe that on a quadrilateral, if one of its vertices is accessible, its opposite vertex must also be accessible. This implies that $L(K_i) = L(K_i) \cap L(K_j)$. The intersection is therefore path-connected by 2.19.

Now consider the case when v_i, v_j are two consecutive vertices on $Lk(v_0, K)$. Observe that in a quadrilateral, two consecutive vertices are accessible if and only if the quadrilateral encloses a strictly convex region. Hence, $Lk(K_i) \cap Lk(K_j)$ consists of all maps $f \in L_0(K)$ such that $St(f(v_0), f(K))$ is strictly convex. By Theorem A, any map $f \in L_0(K) \subset L(K)$ with $St(f(v_0), f(K))$ strictly convex may be connected to the identity element $e \in L(K)$ by a path α in $L(K)$ such that, for each t, $St(\alpha_t(v_0), \alpha_t(K))$ is strictly convex. Let $r: L(K) \to L_0(K)$ be the retraction. Then $r\alpha$ is a path in $L_0(K)$ connecting f to the element e_0. This shows that $L(K_i) \cap L(K_j)$ is path-connected for two consecutive vertices v_i, v_j on $Lk(v_0, K)$. This settles the case for $m_0 = 4$.

We shall now consider the case when $m_0 = 5$. To facilitate the reduction, we need a slightly modified version of the Seifert and Van Kampen theorem. A proof of which will be given in the appendix.

Theorem 3.2 (Seifert and Van Kampen theorem). *Let X be a topological space with a base point x_0. Suppose that X is covered by a collection $\{U_a\}_{a \in A}$ of simply-connected open subsets with $x_0 \in \bigcap_{a \in A} U_a$ and that there exists a subset $J \subset A \times A$ with the properties:*

1. For each $(i, j) \in J$, $U_i \cap U_j$ is path-connected.

2. For any pair $(a, b) \in A \times A$ such that $(a, b) \notin J$, there exists an index $i \in A$ such that both $(a, i), (i, b) \in J$ and $U_a \cap U_b \subset U_i$. Then the space X is also simply connected.

Note that in the above theorem we do not require the collection $\{U_a\}_{a \in A}$ to satisfy the finite intersection property or the path-connectedness for the intersection of each pair of the open sets U_a's.

In our case, the space $L_0(K)$ is covered by five path-connected open subsets $\{L(K_i)\}$, one for each vertex v_i of $Lk(v_0, K)$, such that the element $e_0 \in \bigcap L(K_i)$. Since each subdivision K_i has only $n - 1$ vertices, by the induction hypothesis, each space $L(K_i)$ is simply connected. We shall now let J be the set of all ordered pairs (i, j) of indices such that v_i, v_j are nonconsecutive vertices on $Lk(v_0, K)$. Suppose that we can establish the following two lemmas:

Lemma 3.3. *For each pair of nonconsecutive vertices v_i, v_j on $Lk(v_0, K)$, $L(K_i) \cap L(K_j)$, as a subspace of $L_0(K)$ through the canonical imbedding, is path-connected.*

Lemma 3.4. *For any pair of consecutive vertices* v_s, v_t *on* $\mathrm{Lk}(v_0, K)$, *let* v_i *be the (unique) vertex of* $\mathrm{Lk}(v_0, K)$ *which is nonconsecutive to either* v_s *or* v_t. *Then* $L(K_s) \cap L(K_t) \subset L(K_i)$.

Then by Theorem 3.2, we may conclude the simple-connectedness of $L_0(K)$. Thence, the case for $m_0 = 5$ will also be settled.

Proof of Lemma 3.3. We shall prove this lemma by constructing a path-connected space B (with the help of Theorem A) and a continuous map $j\colon B \to L_0(K)$ such that $j(B) = L(K_i) \cap L(K_j)$.

1. To define the space B, we shall first construct a subdivision K' of S by retriangulating the region $\mathrm{St}(v_0, K)$. B will then be a subspace of $L(K')$.

Now v_i, v_j are two nonconsecutive vertices on $\mathrm{Lk}(v_0, K)$ and $m_0 = 5$. Let v_a be the vertex on $\mathrm{Lk}(v_0, K)$ which is next to both v_i and v_j. Let v_b, v_c be the remaining two vertices on $\mathrm{Lk}(v_0, K)$ such that v_b is next to v_i. Finally, let P be the polyhedral circle $\mathrm{Lk}(v_0, K)$ and Q be the polyhedral circle with the 1-simplices $\langle v_i, v_j \rangle$, $\langle v_j, v_c \rangle$, $\langle v_c, v_b \rangle$ and $\langle v_b, v_i \rangle$. Q is not a subcomplex of K, for $\langle v_i, v_j \rangle \notin K$.

We shall define a subdivision K' of S by requiring K' to agree with K outside the region $\mathrm{St}(v_0, K)$ and on that region (note: it is strictly convex by the hypothesis), we first require $\langle v_a, v_i, v_j \rangle$ to be a 2-simplex of K'. On the remaining part of $\mathrm{St}(v_0, K)$ (i.e., the part enclosed by the polygon Q), we choose an arbitrary point v_0' in the interior of that region, then triangulate it by starring v_0' with respect to Q. Note that this does give a subdivision of S and both P and Q are subcomplexes of K' (in fact, Q becomes $\mathrm{Lk}(v_0', K')$). Also observe that $\mathrm{St}(v_0', K')$ is strictly convex and in K', $m_{v_0'} = 4$.

Now, let B be the subspace of $L(K')$ consisting of those $f \in L(K')$ such that $f(\mathrm{St}(v_0', K'))$ is strictly convex. Observe that Theorem A claims exactly that B is a path-connected space.

2. To define the map $j\colon B \to L_0(K)$, note that for each $f \in B$, $f(P)$ is a polyhedral circle with five vertices, hence, by 2.14, $\mathrm{cor}(f(P)) \neq \varnothing$. Therefore, there exists a map $\overline{f} \in L_0(K)$ such that $\overline{f}(v) = f(v)$ for each vertex $v \neq v_0$ in K and $\overline{f}(v_0) =$ the centroid of $\mathrm{cor}(f(P))$. Now, define $j(f) = \overline{f}$. Clearly, j is a continuous map: $B \to L_0(K)$.

3. We now show that $j(B) \subset L(K_i) \cap L(K_j)$ as subsets in $L_0(K)$, i.e., if $\overline{f} \in L_0(K)$ equals to $j(f)$ for some element $f \in B$, then both $\overline{f}(v_i)$, $\overline{f}(v_j)$ are accessible on $\overline{f}(P)$. Let us first consider $\overline{f}(v_i)$. We want to show that $\overline{f}(v_i) \in CW_{\overline{f}(v_i)}(\overline{f}(P))$. Since $\overline{f}$ agrees with f on all vertices $v \neq v_0$, this is the same as showing that $f(v_i) \in CW_{f(v_i)}(f(P))$.

Now, $\langle v_a, v_i, v_j \rangle$ is a 2-simplex of K', so $f(v_i) \in H_{f(P)}(f(v_a), f(v_j))$ (for this

notation, see 2.1). On the other hand, $f \in B$ implies that $f(Q)$ encloses a convex region, hence, $f(v_i) \in H_{f(P)}(f(v_j), f(v_c)) \cap H_{f(P)}(f(v_c), f(v_b))$. Therefore, $f(v_i) \in CW_{f(v_i)}(f(P))$. A similar argument applies for the accessibility of $\overline{f}(v_j)$ on $\overline{f}(P)$.

4. Finally, we need to show that $j(B) \supset L(K_i) \cap L(K_j)$. Suppose that $\overline{f} \in L_0(K)$ is an element of $L(K_i) \cap L(K_j)$. We now construct a map $f: K' \to S$ by letting $f(v) = \overline{f}(v)$ for each $v \neq v_0$ and $f(v_0') =$ the centroid of $\mathrm{cor}(f(Q))$. Note that the accessibility of both $\overline{f}(v_i)$, $\overline{f}(v_j)$ on $\overline{f}(P)$ forces the 1-simplex $\langle \overline{f}(v_i), \overline{f}(v_j) \rangle$ to lie on the region enclosed by $f(P)$. Hence, f is a well-defined map in $L(K')$. Clearly, $j(f) = \overline{f}$.

To show that f actually belongs to B, we need to show that $f(\mathrm{St}(v_0', K'))$ is strictly convex, i.e., the interior angle of $f(Q)$ at each vertex is less than π. The fact that $f(v_i) = \overline{f}(v_i)$ is accessible on $\overline{f}(P)$ implies that the interior angle of $f(Q)$ at $f(v_b)$ must be less than π. It also implies that $f(v_i) \in H_{f(P)}(f(v_j), f(v_c))$ $= H_{f(Q)}(f(v_j), f(v_c))$, hence, the interior angle of $f(Q)$ at $f(v_j)$ is also less than π. A similar consideration on $f(v_j)$ implies that the interior angles of $f(Q)$ at the other two vertices, namely $f(v_i)$ and $f(v_c)$ are also less than π. This finishes the proof of Lemma 3.3.

Proof of Lemma 3.4. Let the remaining two vertices of $\mathrm{Lk}(v_0, K)$ be denoted by v_p and v_q. Consider an element $f \in L_0(K)$ such that both $f(v_s)$ and $f(v_t)$ are accessible on $\mathrm{Lk}(f(v_0), f(K))$. We shall show that $f(v_i)$ is also accessible on $\mathrm{Lk}(f(v_0), f(K))$. Observe that the accessibility of a vertex w on a pentagon forces the interior angles of the pentagon at the two vertices next to w to be less than π. In our situation, the interior angles of $\mathrm{Lk}(f(v_0), f(K))$ at the vertices $f(v_s)$, $f(v_t)$, $f(v_p)$ and $f(v_q)$ are therefore all less than π. The remaining vertex $f(v_i)$ is clearly accessible on $\mathrm{Lk}(f(v_0), f(K))$.

4. **Proof of the reduced statement.** To facilitate the proof. We shall introduce some more notations. Let v_0 be an inner vertex of a subdivision K. We shall denote by $M(K, v_0)$ the subspace of $L(K)$ consisting of all elements $f \in L(K)$ such that $\mathrm{St}(f(v_0), f(K))$ is strictly convex. Our Theorem A is then clearly a consequence of the following.

Theorem A'. *Let K be a proper subdivision of S, v_0 an inner vertex of K such that $m_0 = 4$ and $\mathrm{St}(v_0, K)$ is strictly convex. The space $M(K, v_0)$ is path-connected.*

We now start our proof of Theorem A' by induction on the number of inner vertices of K. The theorem is vacuously true if there is only one inner vertex. We now consider a proper subdivision K with n inner vertices which has an inner vertex v_0 with $m_0 = 4$ and $\mathrm{St}(v_0)$ strictly convex. The path-connectedness of $M(K, v_0)$ will be proven in the following four steps:

I. We shall show that the space $M(K, v_0)$ is path-connected if K and v_0 satisfy the following three further conditions:

1. K is an indecomposable subdivision.

2. For each inner vertex $v_i \in \text{Lk}(v_0, K)$, $m_i \geq 5$.

3. There exists an inner vertex $v_1 \notin \overline{\text{St}}(v_0)$ such that $m_1 \leq 5$ and v_0, v_1 are a nonbinding pair.

II. We shall show that if K and v_0 satisfy conditions 1 and 2 of Step I, the condition 3 is automatically satisfied.

III. We shall then establish the path-connectedness of $M(K, v_0)$ when K is a decomposable subdivision.

IV. Finally, we shall show that $M(K, v_0)$ is path-connected if there exists an inner vertex $v_i \in \text{Lk}(v_0, K)$ with $m_i \leq 4$.

Step I. If v_i, v_j are any two inner vertices of K such that $v_j \notin \overline{\text{St}}(v_i)$, we shall use $M_{v_j}(K, v_i)$ or simply $M_j(K, v_i)$ to denote the subspace $M(K, v_i) \cap L_{v_j}(K)$ of $L(K)$ where $L_{v_j}(K)$ is the space introduced in Definition 2.5. Note that $M_j(K, v_i)$ is a deformation retract of $M(K, v_i)$, for restricting the deformation $D: L(K) \to L_{v_j}(K)$ described in 2.6 to the subspace $M(K, v_i)$, we clearly get a desired deformation carrying $M(K, v_i)$ onto $M_j(K, v_i)$. In particular, to show the path-connectedness of $M(K, v_i)$, one need only find an inner vertex $v_j \notin \overline{\text{St}}(v_i)$ and show the path-connectedness of $M_{v_j}(K, v_i)$.

Let K, v_0 satisfy the three conditions given in Step I and let v_1 be an inner vertex of K described in condition 3. We shall show the path-connectedness of $M(K, v_0)$ by showing the path-connectedness of the space $M_{v_1}(K, v_0)$.

Similar to Remark 1.2 for the space $L(K)$, we observe that for each $f \in L(K)$, the space $M(K, v_0)$ and $M(f(K), f(v_0))$ are homeomorphic. Hence, applying 1.7 and 1.2 if necessary, we may assume that $\text{St}(v_0, K)$ and $\text{St}(v_1, K)$ are both strictly convex. Then we observe that the splitting of the space $L_{v_1}(K)$ into a union of open subspaces described in 2.16 induces a splitting of the space $M_{v_1}(K, v_0) = \bigcup M(K_i, v_0)$ where each K_i is a subdivision of S obtained by collapsing the vertex v_1 to a vertex v_i on $\text{Lk}(v_1, K)$. Hence, there are at most five such spaces $M(K_i, v_0)$. Each K_i is a proper subdivision with only $n - 1$ vertices and, for an element $f \in M_{v_1}(K, v_0)$, f belongs to $M(K_i, v_0)$ if and only if $f(v_i)$ is accessible on $\text{Lk}(f(v_1), f(K))$.

Let e' be the element of $L_{v_1}(K)$ determined by $e'(v) = v$ for all vertices v of K such that $v \neq v_1$ and $e'(v_1)$ is located at the centroid of the set $\text{cor}(\text{Lk}(v_1, K))$. Note that $e' \in M_{v_1}(K, v_0)$, for $\text{St}(e'(v_0), e'(K)) = \text{St}(v_0, K)$ is strictly convex. Also note that $e' \in M(K_i, v_0)$ for each i, because $\text{St}(e'(v_1), e'(K)) = \text{St}(v_1, K)$ is also strictly convex and therefore, $e'(v_i)$ is accessible on $\text{Lk}(e'(v_1), e'(K))$ for each i. Now by induction hypothesis, each

$M(K_i, v_0)$ is path-connected and $e' \in \bigcap_i M(K_i, v_0)$. Hence, the space $M_{v_1}(K, v_0)$ $= \bigcup_i M(K_i, v_0)$ is also path-connected.

Step II. In the following, let K and v_0 satisfy the conditions 1 and 2 listed in Step I. We shall first give a criterion for a vertex to form a binding pair with v_0.

Lemma 4.1. *For any vertex $v_a \notin \overline{St}(v_0)$ of K, v_a forms a binding pair with v_0 if and only if $\text{Lk}(v_a) \cap \text{Lk}(v_0)$ is disconnected.*

Proof. If v_a, v_0 are a binding pair and $\text{Lk}(v_a) \cap \text{Lk}(v_0)$ is connected, then from the definition of binding pairs, $\text{Lk}(v_a) \cap \text{Lk}(v_0)$ must be a polygonal chain of at least 3 vertices. Consider any vertex v_i of this chain which is not at the end of the chain. v_i must be an inner vertex of K with $m_i = 4$. This contradicts condition 2 of Step I.

The converse follows directly from Definition 1.6.

Lemma 4.2. *K and v_0 also satisfy condition 3 of Step I if there is a vertex $v_p \notin \overline{St}(v_0)$ such that v_0, v_p form a binding pair.*

Proof. Let v_p be such a vertex. By 4.1, $\text{Lk}(v_0) \cap \text{Lk}(v_p)$ is disconnected. Observe that no two vertices which are consecutive on $\text{Lk}(v_0, K)$ can belong to different components of $\text{Lk}(v_0) \cap \text{Lk}(v_p)$, for if there were such vertices, say v_a and v_b, then $\langle v_p, v_a, v_b \rangle$ would have become a pivot triangle in K. This would contradict condition 1 of I. Therefore, $\text{Lk}(v_0) \cap \text{Lk}(v_p)$ must consist of exactly two components; each of which consists of a single vertex, for $\text{Lk}(v_0)$ contains only four vertices. Let v_s, v_t be the two vertices in $\text{Lk}(v_0) \cap \text{Lk}(v_p)$. Let v_h be the unique vertex on $\text{Lk}(v_0)$ such that the region D_p enclosed by the 1-simplices $\langle v_p, v_s \rangle$, $\langle v_s, v_h \rangle$, $\langle v_h, v_t \rangle$ and $\langle v_t, v_p \rangle$ does not contain the vertex v_0.

By the same observation made above, if there is any vertex $v_q \in \text{Int}(D_p)$ such that $\text{Lk}(v_0) \cap \text{Lk}(v_q)$ is disconnected, the set $\text{Lk}(v_0) \cap \text{Lk}(v_q)$ must consist of exactly two components. These two components will have to be the vertices v_s and v_t. Since there are only finitely many vertices in the region D_p, we may assume, by relabelling v_p and D_p if necessary, that for each vertex $v_q \in \text{Int}(D_p)$, $\text{Lk}(v_0) \cap \text{Lk}(v_q)$ is connected.

Now, for each vertex v_k of D_p, let m'_k be the number of 1-simplices of D_p incident on v_k. Note that v_p, v_t, v_h and v_s are the only vertices of D_p lying on $\text{Bd}(D_p)$, we have, by the Combinatorial Lemma 2.17.

$$\sum_{v_i \in \text{Int}(D_p)} (6 - m'_i) = m'_p + m'_s + m'_h + m'_t - 10.$$

Note that m_s' and m_t' both ≥ 3 since they are clearly at least 2 and if one of them equals to 2, v_b will have to be a vertex in $\mathrm{Lk}(v_p)$, and hence, in $\mathrm{Lk}(v_0) \cap \mathrm{Lk}(v_p)$. This contradicts the fact that $\mathrm{Lk}(v_0) \cap \mathrm{Lk}(v_p) = \{v_s, v_t\}$. We also claim $m_p' \geq 3$, for otherwise, $\langle v_t, v_s \rangle \in D_p$ and $\mathrm{St}(v_0, K)$ cannot be strictly convex. Finally, by condition 2 of I, we clearly have $m_b' \geq 4$. Therefore,

$$\sum_{v_i \in \mathrm{Int}(D_p)} (6 - m_i') \geq 3 + 3 + 4 + 3 - 10 = 3.$$

Hence, there exists at least one vertex $v_1 \in \mathrm{Int}(D_p)$ such that $m_1' \leq 5$. Being a vertex in $\mathrm{Int}(D_p)$, v_1 has to be an inner vertex of K and clearly, $m_1 = m_1'$. Furthermore, by the assumption made above, $\mathrm{Lk}(v_0) \cap \mathrm{Lk}(v_1)$ is connected, hence, by 4.1, v_0 and v_1 are a nonbinding pair. Thus, condition 3 of I is satisfied.

The general case in our Step II then follows from the following lemma.

Lemma 4.3. *For the given K and v_0, we can always find an inner vertex $v_p \notin \overline{\mathrm{St}}(v_0)$ with $m_p \leq 5$.*

For if the vertex v_p in the above lemma forms a nonbinding pair with v_0, we just take v_p to be the vertex v_1 in condition 3 of I and there is nothing further to prove. However, if v_p and v_1 are a binding pair, Lemma 4.2 settles the case completely.

Proof of Lemma 4.3. Since K is a proper subdivision, there are only three vertices on $\mathrm{Bd}(S)$. Let e_1, e_2, and e_3 be these three vertices. For each i, we shall let m_{e_i} be the number of 1-simplices of K incident on e_i. Since K and v_0 are assumed to satisfy the conditions 1 and 2 of Step I, we may claim that $\sum_{i=1}^{3} m_{e_i} > 12$. This follows from a case by case study on all the possible subdivisions with $m_{e_i} \leq 12$. Observe that for K to have any inner vertices at all, each m_{e_i} must at least be 3. For the sum $\sum_{i=1}^{3} m_{e_i} \leq 12$, the subdivision K must be one of the following three cases.

a. $\sum m_{e_i} = 9$. This happens if and only if K has exactly one inner vertex.

b. $\sum m_{e_i} = 11$. Note that the number of inner 1-simplices of K incident on each vertex e_i must be respectively 1, 2, 2 (not necessarily in that order). This happens if and only if K has only two inner vertices. The incidence number of these two inner vertices must be 3 or 4.

c. $\sum m_{e_i} = 12$. In this case K may have any number (greater than 2) of inner vertices. But since the numbers of inner 1-simplices incident on the vertices e_i must be either 1, 2, 3 or 2, 2, 2, there must be inner vertices v_a, v_b, v_c of K, each of which is connected to one or two of the e_i's such that the subdivision K is obtained by possibly further subdividing the triangle $\langle v_a, v_b, v_c \rangle$.

Note that in each of these cases, either it is impossible to have a vertex v_0 with $m_0 = 4$ or one of the conditions 1 and 2 of Step I is not satisfied by K and v_0. Hence, we may assume $\Sigma\, m_{e_i} > 12$.

We then apply the Combinatorial Lemma 2.17 to S to get

$$\sum_{v_i \in \text{Int}(S)} (6 - m_i) = 6 + \left(\sum_{i=1}^{3} m_{e_i} \right) - 12 > 6.$$

We now have assumed that $m_0 = 4$ and, for each inner vertex $v_i \in \text{Lk}(v_0, K)$, $m_i \geq 5$. There are at most four such v_i's, hence,

$$\sum_{v_i \text{ inner } \notin \overline{\text{St}}(v_0)} (6 - m_i) > 6 - \sum_{v_i \text{ inner } \in \overline{\text{St}}(v_0)} (6 - m_i) \geq 6 - 2 - 4 = 0.$$

Therefore, there is at least one inner vertex $v_p \notin \overline{\text{St}}(v_0)$ for which $m_p \leq 5$.

Step III. We now show the path-connectedness of the space $M(K, v_0)$ when K is a decomposable subdivision. Suppose $\langle v_a, v_b, v_c \rangle$ is a pivot triangle of K. We shall let K_1 be the subdivision of S which agrees with K outside the region $\langle v_a, v_b, v_c \rangle$ and contains $\langle v_a, v_b, v_c \rangle$ as a 2-simplex. Let K_2 be a subdivision of S which is isomorphic to the triangulation on $\langle v_a, v_b, v_c \rangle$ inherited from K. As we remarked in the proof of 1.4, $L(K)$ is homeomorphic to the Cartesian product $L(K_1) \times L(K_2)$. To be more specific, let $h_1 : K_2 \rightarrow \langle v_a, v_b, v_c \rangle$ be the linear homeomorphism which carries the three vertices e_1, e_2, e_3 of S onto v_a, v_b, v_c respectively. We see that each $f \in L(K)$ corresponds to a pair $(f_1, f_2) \in L(K_1) \times L(K_2)$ as follows: f_1 is the map which agrees with f outside the region $\langle v_a, v_a, v_c \rangle$ and is linear on that region. f_2 is the composite $h_2 \cdot (f | \langle v_a, v_b, v_c \rangle) \cdot h_1$ where h_2 is the unique linear homeomorphism from $\langle f(v_a), f(v_b), f(v_c) \rangle$ onto S carrying the vertices $f(v_a), f(v_b), f(v_c)$ respectively onto e_1, e_2, e_3 of S.

Returning to our proof, we first note that the vertex v_0 cannot be one of the vertices v_a, v_b, v_c, for otherwise, say $v_0 = v_a$, then by the definition of pivot triangle, v_b and v_c must be nonconsecutive vertices on $\text{Lk}(v_0, K)$, hence, $\text{St}(v_0, K)$ cannot be convex. Therefore, v_0 will have to be either inside $\text{Int}\langle v_a, v_b, v_c \rangle$ or completely outside the closed triangle $\langle v_a, v_b, v_c \rangle$.

In the first case, one sees that $\text{St}(f(v_0), f(K))$ is strictly convex if and only if $\text{St}(f_2(v_0), f_2(K_2))$ is strictly convex, for the strict convexity is preserved by the linear homeomorphism h_2. This means that $M(K, v_0)$ is homeomorphic to $L(K_1) \times M(K_2, v_0)$. Hence, $M(k, v_0)$ is path-connected because $L(K_1)$ is path-connected by 2.19 and $M(K_2, v_0)$ is path-connected by the induction hypothesis.

In the second case, i.e., the case $v_0 \notin \langle v_a, v_b, v_c \rangle$, we have similarly $M(K, v_0) = M(K_1, v_0) \times L(K_2)$. Hence, $M(K, v_0)$ is also path-connected. This finishes Step III.

Step IV. We now consider the case when there is an inner vertex $v_i \in$ Lk (v_0, K) with $m_i \le 4$. By Step III, we may assume $m_i \ne 3$, Hence, $m_i = 4$. Let v_a, v_c be the two vertices just next to v_i on Lk (v_0) and v_b be the remaining vertex of Lk (v_0). Since $v_i \in$ Lk (v_0), v_a, v_0, v_c must all be vertices of Lk (v_i). Let v_d be the unique vertex of Lk (v_i) which is not on $\overline{St}(v_0)$. Observe that v_a, v_b, v_c, and v_d form a polyhedral circle in K which is the boundary of the star of the closed 1-simplex $\langle v_0, v_i \rangle$ in K. Let P be this polyhedral circle.

Since St (v_0, K) is strictly convex, the open segment (v_a, v_c) is contained in cor (P). Therefore, we may modify the subdivision K to a subdivision K' of S by eliminating all the simplices of K containing either v_0 or v_i as a vertex and then inserting an 1-simplex $\langle v_a, v_c \rangle$ inside P. P is also a subcomplex of K'.

We shall describe an imbedding of the space $L(K')$ into $M(K, v_0)$ such that the image of the imbedding is a deformation retract of $M(K, v_0)$. This will clearly finish our proof, for the space $L(K')$ is path-connected by 2.19.

For each $f' \in L(K')$, note that cor $(f'(P))$ always contains the open segment $(f'(v_a), f'(v_c))$. As a consequence, for each f', the set $K(f') = $ cor $(f'(P)) \cap \langle f'(v_a), f'(v_c), f'(v_d) \rangle$ is nonempty. We now define an imbedding of $L(K')$ into $M(K, v_0)$ by sending each $f' \in L(K')$ onto a map $f \in L(K)$ to be defined as follows: First let $f(v) = f'(v)$ for all vertices $v \ne v_0$ or v_i and let $f(v_i)$ be located at the centroid of the set $K(f')$. This defines f for all vertices of K except v_0, in particular, f is defined on Lk (v_0, K). Observe that $f(v_i) \in K(f')$ implies that $f(\text{Lk} (v_0, K))$ bounds a strictly convex region in the plane. We then simply define $f(v_0)$ to be the midpoint of the segment $\langle f(v_a), f(v_c) \rangle$. This clearly defines an imbedding of $L(K')$ into $M(K, v_0)$.

Let M be the image of this imbedding of $L(K')$. Then M consists of all those $f \in M(K, v_0)$ such that $f(v_i)$ lies at the centroid of the set $K(f) = $ cor $(f(P)) \cap \langle f(v_a), f(v_c), f(v_d) \rangle$ and $f(v_0)$ is at the midpoint of $\langle f(v_a), f(v_c) \rangle$.

It is not difficult to show that M is a deformation retract of $M(K, v_0)$. We need first define a deformation D_1 which moves, for each $f \in M(K, v_0)$, the vertex $f(v_0)$ to the midpoint of the segment $\langle f(v_a), f(v_c) \rangle$. Note that the centroid of the set $K(f)$ may now be reached by $f(v_i)$. Then let D_2 be the straight line homotopy which carries $f(v_i)$ to the centroid $K(f)$ for each f. The composition, $D_2 D_1$ is a desired deformation.

This finishes Step IV. The proof of Theorem A$'$ is now completed.

5. **Appendix.** We now give a proof to the modified Seifer-Van Kampen theorem (Theorem 3.2) which was used in the paper.

Let $\gamma: I \to X$ be an arbitrary loop in X based at the point x_0. Let $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ be an arbitrary subdivision of the unit interval I such that

the length of each closed subinterval $I_i = [t_{i-1}, t_i]$ is less than the Lebesgue number of the open covering $\{f^{-1}(U_a) \mid a \in A\}$ of I. For each $i = 1, 2, \cdots, n$, choose an open set U_{a_i} such that $\gamma(I_i) \subset U_{a_i}$. Note that each endpoint $\gamma(t_i) \in U_{a_i} \cap U_{a_{i+1}}$ for $i = 1, 2, \cdots, n-1$.

Now for each i, if $(a_i, a_{i+1}) \notin J$, fix a $b_i \in A$ such that both (a_i, b_i) and (b_i, a_{i+1}) belong to J and $U_{a_i} \cap U_{a_{i+1}} \subset U_{b_i}$. In particular, $\gamma(t_i) \in U_{a_i} \cap U_{a_{i+1}} \cap U_{b_i}$. Let δ_i be a positive real number so small that

1. $\delta_i < \frac{1}{2}\min(|t_i - t_{i-1}|, |t_{i+1} - t_i|)$;
2. $\gamma[t_i - \delta_i, t_i + \delta_i] \subset U_{a_i} \cap U_{a_{i+1}} \cap U_{b_i}$.

Then in the original subdivision, drop t_i and insert $t_{i_a} = t_i - \delta_i$, $t_{i_b} = t_i + \delta_i$ and let $U_{a_i}, U_{b_i}, U_{a_{i+1}}$ be the open sets corresponding to the intervals $[t_{i-1}, t_{i_a}]$, $[t_{i_a}, t_{i_b}], [t_{i_b}, t_{i+1}]$. Since there are only finitely many t_i's to consider, we may assume, after relabelling the t_i's if necessary, that for each $i = 1, 2, \cdots, n-1$, $\gamma(t_i) \in U_{a_i} \cap U_{a_{i+1}}$ and $(a_i, a_{i+1}) \in J$.

Now choose a path g_i in $U_{a_i} \cap U_{a_{i+1}}$ for each $i = 1, 2, \cdots, n-1$ joining the point x_0 to the point $\gamma(t_i)$. Let γ_i denote the path in X represented by $\gamma \mid I_i$ for each $i = 1, 2, \cdots, n$. Then each of $\gamma_1 g_1^{-1}, g_1 \gamma_2 g_2^{-1}, g_2 \gamma_3 g_3^{-1}, \cdots,$ $g_{n-2}\gamma_{n-1}g_{n-1}^{-1}, g_{n-1}\gamma_n$ is a closed path, contained in a single open set U_{a_i} and their product in the given order is homotopic to γ in X. Since each of these loops is homotopic to the constant loop as the set U_{a_i} is simply connected, the loop γ is therefore homotopic to the constant loop.

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